Math 1060Q - PreCalculus
Polynomial Division

There are different ways to divide polynomials that go by different names – if you learned a method in another class that you like, then you can stick with it. For this class, we're only going to cover one method that's very similar to long division of numbers you learned in elementary school. First, we'll do out an example, and explain it at the end. Then we'll do a full example, explaining as we go along. Last, we'll just do an example with no explanation that you can follow along with yourself to check if you're understanding correctly.

Example Say we want to divide $x^3 - 23x + 28$ by $x - 4$. We would do the following steps:

$$
\begin{array}{rrrrr}
 & x^2 & +4x & -7 & \\
\cline{2-5}
x-4 \big) & x^3 & & -23x & +28 \\
 & -x^3 & +4x^2 & & \\
\cline{2-3}
 & & 4x^2 & -23x & \\
 & & -4x^2 & +16x & \\
\cline{3-4}
 & & & -7x & +28 \\
 & & & 7x & -28 \\
\cline{4-5}
 & & & & 0
\end{array}
$$

What we've done is to write the dividend (the thing to be divided) inside a "long division sign" and the divisor (the thing dividing) outside, just as we would for numbers. In the dividend, we added a space, since there was no x^2 term (you can think of that space as being like $0x^2$). We then look at the largest degree term of our divisor (the x in the $x - 4$), and think: "what do I need to multiply that by in order to get the largest degree term in my dividend?" (the largest degree term in the dividend is x^3). The answer is x^2. Therefore, we write an x^2 "up top" and multiply every term in our divisor by x^2, writing the answer below the dividend (since $x^2(x - 4) = x^3 - 4x^2$, we write $x^3 - 4x^2$ below our dividend). We then subtract what we've just written from the dividend (that's why it says $-x^3 + 4x^2$, since we are going to subtract $(x^3 - 4x^2)$ and $-(x^3 - 4x^2) = -x^3 + 4x^2$. The result is $4x^2 - 23x + 28$, but we'll only write the $4x^2 - 23x$ part since that's what we'll need for the next step. Now we take our new dividend $(4x^2 - 23x + 28)$, look at its largest degree term $(4x^2)$, and think: "what do I need to multiply the largest degree term in my divisor by in order to get the largest degree term in my dividend?" (i.e., what do I need to multiply x by in order to get $4x^2$?) The answer is $4x$. So, repeating the above process, we write a $4x$ up top, multiply $x - 4$ by $4x$, we get $4x^2 - 16x$, and we subtract that (i.e, we add $-4x^2 + 16$) from our dividend, and end up with $-7x + 28$. Now we see that if we multiply $x - 4$ by -7 then we'll match the $-7x$, so we write a -7 up top, multiply $x - 4$ by -7, write that down below, subtract it, and we're left with 0. Therefore, we're done, and our answer is that

$$(x^3 - 23x + 28) \div (x - 4) = x^2 + 4x - 7$$

or in other words,

$$(x - 4)(x^2 + 4x - 7) = x^3 - 23x + 28.$$

Now let's do another example, breaking it down into steps.

Example Let's divide $2x^4 - 3x^3 + 2x^2 + x - 3$ by $2x - 1$. We first set up the long division, adding empty spaces if necessary (not necessary in this case).

$$
\begin{array}{rl}
 & \\
\cline{2-2}
2x-1 \big) & 2x^4 - 3x^3 + 2x^2 + x - 3
\end{array}
$$

Then we begin, by figuring out what we need to multiply $2x$ by in order to get $2x^4$ (x^3 is what works):

$$
\begin{array}{rrrrrr}
 & & x^3 & & & \\ \cline{2-6}
2x-1 \big) & 2x^4 & -3x^3 & +2x^2 & +x & -3
\end{array}
$$

We then multiply $2x-3$ by x^3, and take its negative because we're going to subtract it from the dividend:

$$
\begin{array}{rrrrrr}
 & & x^3 & & & \\ \cline{2-6}
2x-1 \big) & 2x^4 & -3x^3 & +2x^2 & +x & -3 \\
 & -2x^4 & +x^3 & & & \\ \cline{2-3}
\end{array}
$$

We do the subtraction (since we added a negative sign in the previous step, we now just need to add the two lines). If the largest term doesn't cancel out and go away, you'll know you made a mistake somewhere.

$$
\begin{array}{rrrrrr}
 & & x^3 & & & \\ \cline{2-6}
2x-1 \big) & 2x^4 & -3x^3 & +2x^2 & +x & -3 \\
 & -2x^4 & +x^3 & & & \\ \cline{2-3}
 & & -2x^3 & +2x^2 & &
\end{array}
$$

Now we repeat, with a new dividend and a new largest term ($-2x^3$). Find what we need to multiply $2x$ by to get $-2x^3$ (must be $-x^2$):

$$
\begin{array}{rrrrrr}
 & & x^3 & -x^2 & & \\ \cline{2-6}
2x-1 \big) & 2x^4 & -3x^3 & +2x^2 & +x & -3 \\
 & -2x^4 & +x^3 & & & \\ \cline{2-3}
 & & -2x^3 & +2x^2 & &
\end{array}
$$

Multiply $-x^2$ by $2x-1$, and take its negative:

$$
\begin{array}{rrrrrr}
 & & x^3 & -x^2 & & \\ \cline{2-6}
2x-1 \big) & 2x^4 & -3x^3 & +2x^2 & +x & -3 \\
 & -2x^4 & +x^3 & & & \\ \cline{2-3}
 & & -2x^3 & +2x^2 & & \\
 & & 2x^3 & -x^2 & & \\ \cline{3-4}
\end{array}
$$

Add the two lines:

$$
\begin{array}{rrrrrr}
 & & x^3 & -x^2 & & \\ \cline{2-6}
2x-1 \big) & 2x^4 & -3x^3 & +2x^2 & +x & -3 \\
 & -2x^4 & +x^3 & & & \\ \cline{2-3}
 & & -2x^3 & +2x^2 & & \\
 & & 2x^3 & -x^2 & & \\ \cline{3-4}
 & & & x^2 & +x &
\end{array}
$$

And repeat again. We'll just finish up this example, and then make a few notes about how it's different from the previous.

$$
\begin{array}{r}
x^3 \quad - x^2 + \tfrac{1}{2}x + \tfrac{3}{4} \\
2x-1 \,\big)\overline{\; 2x^4 - 3x^3 + 2x^2 \quad + x - 3} \\
\underline{-\,2x^4 \quad + x^3} \qquad\qquad\qquad\qquad \\
-\,2x^3 + 2x^2 \qquad\qquad \\
\underline{2x^3 \quad - x^2} \qquad\qquad \\
x^2 \quad + x \qquad \\
\underline{-\,x^2 + \tfrac{1}{2}x} \qquad \\
\tfrac{3}{2}x \; - 3 \\
\underline{-\,\tfrac{3}{2}x + \tfrac{3}{4}} \\
-\,\tfrac{9}{4}
\end{array}
$$

One difference you'll notice is that we had to introduce fractions. This is perfectly fine, and in fact, it's only in math textbooks that you'll find long division working out so nicely with only integers. In general, of course, fractions and funny decimals are much more common. But, you'll want to brush up on your arithmetic with fractions if you're rusty!

Another, more important, difference is that we didn't end up with 0 at the end, we ended up with $-\frac{9}{4}$. This means that $2x-1$ does not go into $2x^4-3x^3+2x^2+x-3$ perfectly, and there is a remainder of $-\frac{9}{4}$ (much like if you divide 13 by 5, you end up with a remainder of 3). We would say then that $2x-1$ is not actually a factor of $2x^4-3x^3+2x^2+x-3$, but we can write:

$$\left[(2x-1)(x^3-x^2+\frac{1}{2}x+\frac{3}{4})\right]-\frac{9}{4}=2x^4-3x^3+2x^2+x-3$$

(This is similar to how we would write $5\cdot 2+3=13$.)

We include one more example, without explanation, that you can try to follow along yourself.

Example Dividing $6x^3+12x^2+7$ by $2x+2$:

$$
\begin{array}{r}
3x^2 + 3x - 3 \\
2x+2 \,\big)\overline{\; 6x^3 + 12x^2 \qquad + 7} \\
\underline{-\,6x^3 \;\; - 6x^2} \qquad\qquad \\
6x^2 \qquad\qquad \\
\underline{-\,6x^2 - 6x} \qquad \\
-\,6x + 7 \\
\underline{6x + 6} \\
13
\end{array}
$$